Singular behavior of ${}^1P_1^{+-}$ quarkonium and positronium annihilation decays and relevance of relative energy

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Abstract. Using a four dimensional approach, we show that the singularities for small gluon momenta, which arise in the usual three dimensional treatment of the annihilation decay, disappear if all poles in the relative energy are taken into account correctly in the integration. We obtain an explicit formula for the decay width which involves a non-locality originating from the kinetic energy. We calculate not only the familiar logarithmic dependence on the binding energy, but also the constant to be added to the logarithm. The logarithmic term agrees with previous values in the literature. In QCD the constant turns out to be quite small, but only because there is an almost perfect cancellation between the tree graph and a non-abelian loop graph which contributes to the decay amplitude to the same order.

1 Introduction

It has been known for a long time that the usual non-relativistic approach for calculating the decay width of the ${}^1P_1^{+-}$-state into three gluons, as well as the annihilation of ${}^3P_1^{++}$ into a gluon and light quark–antiquark pairs, has infrared divergences originating from soft gluon momenta. In the original paper by Barbieri, Gatto, and Remiddi [1] the problem was avoided by introducing a fictituous width for the decay into a real and a virtual photon (or gluon). Their result was ($R_{21}(r)$ is the radial wave function)

$$\Gamma\left({}^1P_1^{+-} \to 3 \text{ gluons}\right) \tag{1.1}$$
$$= \frac{5}{18}\frac{8}{\pi}\frac{\alpha_s^3}{m^4}|R'_{21}(0)|^2 \left[\ln\frac{m}{|E|} + \text{const.}\right].$$

Actually only the singular logarithmic dependence on the binding energy could thus be calculated, while the constant above was simply ignored. This is a shortcoming because such a constant, even if not particularly large, could be quite important since the logarithm is not large in QCD and even only about 12.6 in QED.

In a series of papers Bodwin, Braaten, Lepage [2] and collaborators clarified some aspects of the problem by applying the methods of non-relativistic QCD (NRQCD) [3]. The quarkonium state is not only made up of quark and antiquark ($Q\bar{Q}$) but contains also Fock state components with quark–antiquark–glue ($Q\bar{Q}g$) etc. Although this admixture of glue is suppressed with respect to the dominant $Q\bar{Q}$-state, it contributes to the decay with the same order. The reason is that the quark–antiquark annihilation in the

$Q\bar{Q}$-state is also suppressed, because the $Q\bar{Q}$ wave function at the origin vanishes in the P-state. The $Q\bar{Q}$ in the $Q\bar{Q}g$ component, on the other side, can be in an S-state, thus compensating the suppression of the Fock state component. The divergences found previously are canceled and the part with the logaritmic dependence on the binding energy can be calculated; the constant which accompanies the logarithm remains, however, unknown.

In a further paper on the subject the present author [4] calculated the glue content in heavy quarkonia in order to obtain more definite statements. Although this calculation was successful and gave reasonable results, it turned out that it was, at least up to now, not helpful for calculating the decay width.

In the present work we go back to a more fundamental formalism. It involves the four dimensional T-matrix for the annihilation of the quark–antiquark pair into gluons, as well as the four dimensional Bethe–Salpeter wave function, taken in the approximation coming from the static Salpeter equation. Both quantities depend on the relative energy $p^0 = (p_1 - p_2)^0/2$ of the quark–antiquark pair. The usual non-relativistic approach is equivalent to considering only the poles of the BS wave function when integrating over p^0. The p^0-dependence of the T-matrix is, however, ignored by fixing the quarks on mass shell there. As will be discussed in the following, this procedure is legitimate in most cases, like decays of S-wave states, or decays into two real gluons. It fails, however, for the decay of the ${}^1P_1^{+-}$-state into three gluons if one of the gluons is soft.

The contributions from the poles in the T-matrix are only of relevance if one of the gluons is soft. In this region, however, they are crucial, because they cancel the singularities in the leading term. Thus the correct consideration of

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all poles gives an expression which is free of divergences. Another phenomenon which was already found in [4] arises in the soft region: A non-abelian graph, where a soft transversal gluon splits into two Coulomb gluons, while the latter couple to quark and antiquark, contributes to the same order as the tree graph. The reason is that the direct coupling of a soft transversal gluon to a non-relativistic quark is also suppressed.

Our result is not directly expressible by the derivative of the radial wave function at the origin, because it contains expressions of the form $k+|E|+\mathbf{p}^2/m$ in the denominator, with k a gluon energy, E the binding energy, and $\mathbf{p}^2/m$ the kinetic energy. The presence of the kinetic energy introduces a non-locality. For a Coulomb potential one can carry through the calculation analytically till the end. We agree with the formulae in the literature as far as the logarithmic dependence on the binding energy is concerned. Beyond this we can calculate the constant term to be added to the logarithm. For the decay of the singlet P-state of positronium into three photons (where the non-abelian contribution is of course absent) our constant differs from the one given by Alekseev [5]. In the non-abelian case the resulting constant is very small, but only due to a miraculous almost total cancellation between large terms originating from the tree graph, the non-abelian loop graph, and the mixing term.

In our approach no $Q\bar{Q}g$ component in the wave function shows up at all. All contributions come from the four dimensional $Q\bar{Q}$ wave function alone. Nevertheless one may reinterpret our result in terms of the non-relativistic picture and the presence of a quark–antiquark–glue component in the wave function.

In Sect. 2 we discuss the situation and derive the general formalism. In Sect. 3 we specialize to a Coulomb wave function and present an analytic formula. In Sect. 4 we discuss the experimental status and future applications.

2 Four dimensional versus three dimensional approach

Consider an annihilation decay of a quarkonium state in the rest frame into three (or analogously into two) gluons. In the familiar three dimensional approach one proceeds as follows. The quark energies p_1^0 and p_2^0 in the T-matrix for $Q\bar{Q} \to$ gluons are taken on mass shell and the quark momenta smeared with the Schrödinger wave function. The amplitude, with the gluon momenta defined as in Fig. 1, then becomes

$$\langle \mathbf{k}_1^a \mathbf{k}_2^b \mathbf{k}_3^c | K \rangle \tag{2.1}$$
$$= \frac{\mathrm{i}(2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3 - K)}{\sqrt{(2\pi)^3 m}} \int T(\mathbf{p}) \tilde{\psi}(\mathbf{p}) \mathrm{d}^3 p,$$

with $p^\mu = (p_1 - p_2)^\mu/2$ the relative momentum. The dependence of the T-matrix on the gluon momenta $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$, polarization vectors $\epsilon_1, \epsilon_2, \epsilon_3$, and color indices a, b, c has been suppressed. We used the standard covariant normalization of states and relativistic normalization of T, the Schrödinger wave function is normalized to 1. The T-matrix

for quark–antiquark into gluons is a slowly varying function of $\mathbf{p}$, with a scale set by the quark mass m, while the wave function $\tilde{\psi}(\mathbf{p})$ is non-relativistic, i.e. dominated by soft momenta of the order of $\alpha_s m$. Therefore one can expand the T-matrix around $\mathbf{p} = 0$. For S-states it is sufficient to consider the leading term $T(0)$, the remaining integral gives the wave function in position space at the origin,

$$T(0) \int \tilde{\psi}(\mathbf{p}) \mathrm{d}^3 p = T(0)(2\pi)^{3/2} \psi(0). \tag{2.2}$$

For P-states (with magnetic quantum number m) one has to expand up to first order in $\mathbf{p}$, resulting in

$$[(\nabla_\mathbf{p})^n T(\mathbf{p})]\Big|_{\mathbf{p}=0} \int p^n \tilde{\psi}(\mathbf{p}) \mathrm{d}^3 p$$
$$= -\mathrm{i}[(\nabla_\mathbf{p})^n T(\mathbf{p})]\Big|_{\mathbf{p}=0} Y_{1m}(\mathbf{e}_{(n)})(2\pi)^{3/2} R'(0), \tag{2.3}$$

with $R(r)$ the radial wave function.

For decays into two gluons, which give the leading contribution for states with positive charge conjugation, i.e. $^1S_0^{-+}$ or $^3P_J^{++}$ for $J = 0, 2$ ($^3P_1^{++}$ cannot decay into two real gluons) there are no problems. Both gluons have to be hard for kinematical reasons. Therefore the denominator of the quark propagator in $T(\mathbf{p})$ cannot vanish and no singularities can arise.

Let us next discuss the three gluon decay shown in Fig. 1 which is the leading contribution for annihilation decays of states with negative charge conjugation, i.e. $^3S_1^{--}$ or $^1P_1^{+-}$.

With the quark energies taken on mass shell, and the quark momenta considered only up to first order, the denominators of the two quark propagators become (we always use $k_j \equiv |\mathbf{k}_j|$ in the following)

$$(p_1 - k_1)^2 - m^2 + \mathrm{i}\epsilon = -2mk_1 + 2\mathbf{p}_1\mathbf{k}_1 + \mathrm{i}\epsilon,$$
$$(k_2 - p_2)^2 - m^2 + \mathrm{i}\epsilon = -2mk_2 + 2\mathbf{p}_2\mathbf{k}_2 + \mathrm{i}\epsilon. \tag{2.4}$$

This is the well known reason for the possible appearance of divergences which can arise from small $\mathbf{k}_1$ or $\mathbf{k}_2$.

Let us now view the situation in the framework of a four dimensional treatment of the three gluon decay, focusing on the problematics for the P-state. The amplitude in Fig. 1 has the form

$$\langle \mathbf{k}_1^a \mathbf{k}_2^b \mathbf{k}_3^c | K \rangle = -\frac{(2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3 - K)}{\sqrt{(2\pi)^5 m}} \tag{2.5}$$
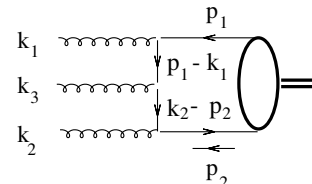$$\times \int \mathrm{Tr}[T(\mathbf{p}, p^0) \Gamma(\mathbf{p})]$$

Fig. 1. The lowest order contribution to the three gluon decay

$$\times \left\{ (|E| + \mathbf{p}^2/m)\, \tilde{\psi}(\mathbf{p}) \right\}$$
$$/ \left\{ (p^0 + |E|/2 + \mathbf{p}^2/2m - \mathrm{i}\epsilon) \right.$$
$$\left. \times (p^0 - |E|/2 - \mathbf{p}^2/2m + \mathrm{i}\epsilon) \right\}\, \mathrm{d}p^0\, \mathrm{d}^3p.$$

As before, T denotes the T-matrix for the annihilation of the quark–antiquark pair into gluons. $\Gamma(\mathbf{p})$ is the spin wave function of the bound state. For a singlet state it reads $\Gamma(\mathbf{p}) = \Lambda(\mathbf{p}_1)(\gamma^5/\sqrt{2})\Lambda^c(\mathbf{p}_2)$. Here $\Lambda(\mathbf{p}_k)$ are the projectors to positive energy states; in our case we need them only up to order $\mathbf{p}_k$, where they read

$$\Lambda(\mathbf{p}_1) = \frac{1+\gamma^0}{2} + \frac{\boldsymbol{\alpha}\mathbf{p}_1}{2m},$$
and $\qquad\qquad\qquad\qquad$ (2.6)
$$\Lambda(\mathbf{p}_2)^c = \frac{1-\gamma^0}{2} - \frac{\boldsymbol{\alpha}\mathbf{p}_2}{2m}.$$

Finally $\tilde{\psi}(\mathbf{p})$ is again the Schrödinger wave function, while E is the binding energy. We used the well known p^0-dependence of the Bethe–Salpeter wave function as it arises from the static approximation to the BS equation.

We first discuss the trace in the numerator. For the graph in Fig. 1 it is

$$\mathrm{Tr}\left\{ \Gamma(\mathbf{p})\gamma^{n_2}[\not{k}_2 - \not{p}_2 + m]\gamma^{n_3}[(\not{p}_1 - \not{k}_1) + m]\gamma^{n_1} \right\}.$$
(2.7)

This is only needed in zeroth order of $\mathbf{p}$ for S-states, and in first order for P-states. The zeroth order of the trace contains factors $\ldots (k_2 - \mathbf{k}_2\,\boldsymbol{\gamma})\ldots(k_1 + \mathbf{k}_1\,\boldsymbol{\gamma})\ldots$; therefore there is no trouble for the three gluon decay of S-states; the factors of the numerator cancel the singularities in $\mathbf{k}_1$ and $\mathbf{k}_2$ of the denominator.

The calculation of the order $\mathbf{p}$ term of the trace is rather easy due to a simplification which comes from the $\mathbf{p}$-dependence of the projectors $\Lambda(\mathbf{p})$ and $\Lambda^c(-\mathbf{p})$. Together with the other factors, it leads to anticommutators of γ-matrices. The order $\mathbf{p}$ term of the trace becomes

$$4\mathrm{i}\left\{ p^{n_2}k_1^n \epsilon_{n_1 n_3 n} + p^{n_1}k_2^n \epsilon_{n_2 n_3 n} \right\}.$$
(2.8)

Now there are no factors which contain both k_1 and k_2. Therefore some of the singularities of the denominator remain uncancelled and lead to the well known divergence in the calculation of the decay of $^1P_1^{+-}$ into three gluons. Note that the trace (2.8) is independent of the zero components p_1^0 and p_2^0. This will facilitate the discussion below.

We next have to perform the integration over p^0 in (2.5). The usual approach is equivalent to fixing the energies p_1^0 and p_2^0 on mass shell in the T-matrix, which means putting $p^0 = |E|/2 + \mathbf{p}^2/(2m)$ in the first, and $p^0 = -|E|/2 - \mathbf{p}^2/(2m)$ in the second quark propagator. Only the p^0-dependence from the Salpeter wave function is considered and the p^0-integration performed; thus we end up with the simple formula (2.1). In this way one has made three approximations.

(1) The values of p^0 in the T-matrix have been fixed by the on shell prescription instead of using the residue at the pole of the wave function.

(2) The contributions from the remaining poles have been neglected.
(3) For P-states, where one has to expand up to order $\mathbf{p}$, the $\mathbf{p}$-dependence of the projection operators Λ has been ignored.

Let us come to the correct p^0-integration in the four dimensional formula (2.5). The Salpeter wave function has a pole ϵ_w^+ in the upper half plane, and a pole ϵ_w^- in the lower half plane. There are four more poles from the two propagators in T, which we denote by $\epsilon_1^\pm$, and $\epsilon_2^\pm$. Altogether the poles are

$$\epsilon_\mathrm{w}^\pm = \mp\left(|E|/2 + \mathbf{p}^2/(2m) - \mathrm{i}\epsilon\right), \qquad (2.9)$$
$$\epsilon_1^\pm = -m + |E|/2 + k_1 \mp \left(\sqrt{m^2 + (\mathbf{k}_1 - \mathbf{p})^2} - \mathrm{i}\epsilon\right),$$
$$\epsilon_2^\pm = m - |E|/2 - k_2 \mp \left(\sqrt{m^2 + (\mathbf{k}_2 + \mathbf{p})^2} - \mathrm{i}\epsilon\right).$$

We have to calculate the following integral where the first four factors come from the quark propagators in the T-matrix, the last two from the four dimensional wave function

$$I = -\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^\infty \left\{ (|E| + \mathbf{p}^2/m) \right\} \qquad (2.10)$$
$$/ \left\{ (p^0 - \epsilon_2^+)(p^0 - \epsilon_2^-)(p^0 - \epsilon_1^+) \right.$$
$$\left. \times (p^0 - \epsilon_1^-)(p^0 - \epsilon_\mathrm{w}^+)(p^0 - \epsilon_\mathrm{w}^-) \right\} \mathrm{d}p^0.$$

We perform the p^0-integration by closing the integration contour in the upper half plane (which, temporarily, introduces an apparent asymmetry between quark and antiquark). There are three contributions originating from the three poles at ϵ_w^+, ϵ_1^+, ϵ_2^+:

$$I = -(|E| + \mathbf{p}^2/m)\left(\frac{1}{D_\mathrm{w}} + \frac{1}{D_1} + \frac{1}{D_2} \right). \qquad (2.11)$$

The denominators which appear here are

$$D_\mathrm{w} = \prod_n{}' (\epsilon_\mathrm{w}^+ - \epsilon_n), \quad D_1 = \prod_n{}' (\epsilon_1^+ - \epsilon_n),$$
$$D_2 = \prod_n{}' (\epsilon_2^+ - \epsilon_n), \qquad (2.12)$$

where the products $\prod_n{}'$ run over the five remaining poles.

If all three gluons are hard, the contribution $1/D_\mathrm{w}$ from the pole ϵ_w^+ is dominant because D_w contains the small ultrasoft difference $\epsilon_\mathrm{w}^+ - \epsilon_\mathrm{w}^- = -(|E| + \mathbf{p}^2/m)$. This cancels the corresponding factor in front of (2.11). All the other differences appearing in the three denominators are hard for hard gluons; therefore the contributions $1/D_1$ and $1/D_2$ are suppressed compared to the leading term $1/D_\mathrm{w}$ by a relative factor of the order $(|E|/m + \mathbf{p}^2/m^2)$. The momentum p_2 is on mass shell on the pole ϵ_w^+, while p_1 is slightly off shell. One may, however, also put p_1 on mass shell in D_w, which corresponds to replacing $\epsilon_\mathrm{w}^+ - \epsilon_1^\pm$ by $\epsilon_\mathrm{w}^- - \epsilon_1^\pm$. This again only causes a relative error of order

$|E|/m$ and leads back to the old formula from the three dimensional treatment.

The situation is more subtle if one of the gluons is soft or ultrasoft. The other two are then, of course, hard.

We start with the case where $\mathbf{k}_3 \to 0$, i.e. when the quark propagators in the T-matrix become identical and create a double pole. Both D_1 and D_2 become singular for $\mathbf{k}_3 \to 0$ which is due to the factor

$$\epsilon_1^+ - \epsilon_2^+ = -k_3 - \sqrt{m^2 + (\mathbf{k}_1 - \mathbf{p})^2}$$
$$+ \sqrt{m^2 + (\mathbf{k}_2 + \mathbf{p})^2} \qquad (2.13)$$

(we used $k_1 + k_2 + k_3 = 2m - |E|$). This factor enters, however, with opposite signs in D_1 and D_2, while all the other factors become identical for $\mathbf{k}_3 \to 0$. Therefore there is no singularity for $\mathbf{k}_3 \to 0$ in the sum.

For $\mathbf{k}_1 \to 0$ the first propagator gets near to the mass shell. This behavior shows up only in D_w and is due to the difference

$$\epsilon_\mathrm{w}^+ - \epsilon_1^- = -|E| - \mathbf{p}^2/(2m) + m - k_1$$
$$- \sqrt{m^2 + (\mathbf{k}_1 - \mathbf{p})^2}$$
$$\to -(k_1 + |E| + \mathbf{p}^2/m) \text{ for } \mathbf{k}_1 \to 0. \quad (2.14)$$

Finally we look at the limit $\mathbf{k}_2 \to 0$, where the second quark propagator becomes on shell. In D_w there is a difference which becomes small, namely

$$\epsilon_\mathrm{w}^+ - \epsilon_2^+ = -\mathbf{p}^2/(2m) - m + k_2 + \sqrt{m^2 + (\mathbf{k}_2 + \mathbf{p})^2}$$
$$\to k_2 \text{ for } \mathbf{k}_2 \to 0. \qquad (2.15)$$

In D_2 there are two differences which become small, namely $\epsilon_2^+ - \epsilon_\mathrm{w}^+ \to -k_2$, i.e. the negative of the term above, and

$$\epsilon_2^+ - \epsilon_\mathrm{w}^- = -|E| - \mathbf{p}^2/(2m) + m - k_2$$
$$- \sqrt{m^2 + (\mathbf{k}_2 + \mathbf{p})^2}$$
$$\to -(k_2 + |E| + \mathbf{p}^2/m) \text{ for } \mathbf{k}_2 \to 0. \quad (2.16)$$

In the sum the two poles at $\mathbf{k}_2 = 0$ cancel, while the ultrasoft term $-(k_2 + |E| + \mathbf{p}^2/m)$ remains.

Summarizing we found the following result: The contributions of $1/D_1$ and $1/D_2$ are negligible if all gluons are hard. They are also negligible for $\mathbf{k}_3 \to 0$ because the poles cancel in the sum, as well as for $\mathbf{k}_1 \to 0$, because they are finite in the latter limit. For $\mathbf{k}_2 \to 0$ finally, the poles at $\mathbf{k}_2 = 0$ in $1/D_\mathrm{w}$ and $1/D_2$ cancel, while the pole (2.16) in $1/D_2$ survives. In total one thus obtains

$$\frac{1}{D_\mathrm{w}} + \frac{1}{D_1} + \frac{1}{D_2} = \frac{1 + O(|E|/m + \mathbf{p}^2/m^2)}{D}, \qquad (2.17)$$

where D is obtained from D_w by replacing $\epsilon_1^+ - \epsilon_2^+$ by $\epsilon_\mathrm{w}^- - \epsilon_2^+$. We may also replace $\epsilon_\mathrm{w}^+ - \epsilon_2^-$ by $\epsilon_\mathrm{w}^- - \epsilon_2^-$ because this difference is always hard. This restores the manifest symmetry between quark and antiquark.

The spatial momentum $\mathbf{p}$ is soft in the bound state described by the wave function $\tilde{\psi}(\mathbf{p})$. Therefore one can put $\mathbf{p} = 0$ (as well as $E = 0$) in all terms where $\mathbf{p}$-dependent terms are added to hard terms. In the two critical factors, i.e. in those which become small for small $\mathbf{k}_1$ or $\mathbf{k}_2$ respectively, one can use the limits in the second lines of (2.14) and (2.16). This finally leads to the following simple form for the integral I in (2.10):

$$I = \frac{1}{4m^2} \frac{1 + O(|E|/m + \mathbf{p}^2/m^2)}{(k_1 + |E| + \mathbf{p}^2/m)(k_2 + |E| + \mathbf{p}^2/m)}. \quad (2.18)$$

The apparent singularities for $\mathbf{k}_1 \to 0$ and $\mathbf{k}_2 \to 0$, which showed up in the three dimensional approach have turned out to be spurious. They disappear if all the poles in the relative energy p^0 are considered in the p^0-integration. For $j = 1, 2$ there remain, however, denominators of the form $k_j + |E| + \mathbf{p}^2/m$ which become small for small k_j and enforce to keep a finite binding energy in order to avoid a divergence. Because the binding energy E and the kinetic energy $\mathbf{p}^2/m$ are of the same order, one must, however, also keep the term $\mathbf{p}^2/m$ for consistency. This leads to a non-local dependence upon the wave function.

We have to multiply (2.18) with the trace (2.8) and the remaining contributions from the expression (2.5). We consider only the first term of the trace (2.8) and multiply by two, because anyhow the other permutations of the gluons have to be added. Since this term contains a k_1^n in the numerator one can drop the ultrasoft terms in the first factor in (2.18). This finally leads to

$$\langle \mathbf{k}_1^a \mathbf{k}_2^b \mathbf{k}_3^c | K \rangle_\mathrm{tree}$$
$$= \left(\frac{d^{abc}}{4\sqrt{N}} \right) \mathrm{i}(2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3 - K)$$
$$\times \frac{g^3 [\epsilon_1 \times \epsilon_3] \cdot \mathbf{k}_1}{\pi^{3/2} m^{3/2} k_1} \int \frac{(\epsilon_2 \cdot \mathbf{p}) \, \tilde{\psi}(\mathbf{p})}{k_2 + |E| + \mathbf{p}^2/m} \, \mathrm{d}^3 p$$
$$+ \text{permutations.} \qquad (2.19)$$

The permutations are the remaining five permutations of the three gluons. The factor $(d^{abc}/(4\sqrt{N}))$ in front is the color factor, with $N = 3$ the number of colors. For the decay of the singlet P-state of positronium into three gammas it has to be replaced by 1, while g is replaced by e.

In QED (2.19) would in fact be the leading contribution to the decay amplitude. In QCD we encounter, however, the same phenomenon which we found already in [4] when calculating the admixture of the quark–antiquark–glue component to the wave function. For the case that $\mathbf{k}_2$, say, is ultrasoft, the loop graph in Fig. 2, where the dashed lines denote Coulomb gluons, contributes to the same order as the tree graph in Fig. 1 (although it will not lead to a logarithmic dependence upon the binding energy).

The reason is that the tree graph is suppressed by the coupling of a transverse gluon to a non-relativistic quark. The loop graph, on the other hand, is suppressed by two more coupling constants, but this is compensated because the coupling to the quarks via Coulomb gluons is not suppressed. Other loop graphs are not relevant, because they do not have a sufficient number of denominators which can become small simultaneously. The same is true for higher
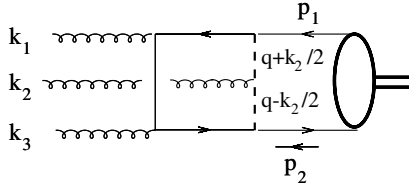
Fig. 2. The loop graph for the three gluon decay. Note that the indices of the gluons have been chosen such that the dominant contribution comes from soft k_2

order corrections in the wave function, e.g. exchange of a transverse gluon. Also the graph in Fig. 2 is only relevant in the region of ultrasoft $\mathbf{k}_2$. This simplifies the calculation considerably.

If one first integrates over q^0 in Fig. 2 one finds a similar situation as before. Spurious poles in k_1 and k_3 which show up at some residues cancel in the sum. Only the variable k_2 appears in a denominator which becomes small but finite for ultrasoft k_2. After having performed the q^0-integration, the whole p^0-dependence is in the BS wave function and the p^0-integration can be immediately performed. Finally one can do the integration over d^3q which is simplified by the fact that one can drop the k_2 in the Coulomb propagators. The resulting expression is

$$
\langle \mathbf{k}_1^a \mathbf{k}_2^b \mathbf{k}_3^c | K \rangle_{\mathrm{loop}}
$$

$$
= \left(\frac{d^{abc}}{4\sqrt{N}} \right) \mathrm{i}(2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3 - K)
$$

$$
\times \frac{g^5 N [\epsilon_1 \times \epsilon_3] \cdot \mathbf{k}_1}{2(2\pi)^{5/2}\sqrt{m}}
$$

$$
\times \int \frac{(\epsilon_2 \cdot \mathbf{p})}{p^3} f\left(\frac{\sqrt{m(k_2 + |E|)}}{p} \right) \tilde{\psi}(\mathbf{p}) \, \mathrm{d}^3 p
$$

$$
+\text{permutations}, \tag{2.20}
$$

with

$$
f(x) = \frac{\pi}{2} - \frac{x}{1 + x^2} - \arctan x. \tag{2.21}
$$

3 Three gluon (or 3 γ) annihilation of the $^1P_1^{+-}$-state

Due to the non-local dependence on the momentum $\mathbf{p}$ in (2.19) and (2.20) one cannot simply express the result by the derivative of the wave function at the origin. Instead one has to perform the p-integration explicitly. We carry this through for the case of the lowest Coulomb P-state wave function which reads (we drop an irrelevant factor i)

$$
\tilde{\psi}(\mathbf{p}) = \frac{8\sqrt{2}\sqrt{m|E|}}{\sqrt{\pi}} R'_{21}(0) \frac{|\mathbf{p}|}{(|\mathbf{p}|^2 + m|E|)^3} Y_{1m}(\hat{\mathbf{p}}), \tag{3.1}
$$

with $|E| \equiv |E_2| = (C_F \alpha_{\mathrm{s}})^2 m/16$ and $C_F = (N^2 - 1)/(2N) = 4/3$. To make contact with the literature we have split off the derivative of the wave function at the origin, $R'_{21}(0) = 2(m|E|)^{5/4}/\sqrt{3}$. The p-integration in (2.19) and (2.20) can now be performed and we obtain

$$
\langle \mathbf{k}_1^a \mathbf{k}_2^b \mathbf{k}_3^c | K \rangle_{\mathrm{Coulomb}}
$$

$$
= \left(\frac{d^{abc}}{4\sqrt{3}} \right) \mathrm{i}(2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3 - K)
$$

$$
\times [\epsilon_1 \times \epsilon_3] \cdot \mathbf{k}_1 \, Y_{1m}(\epsilon_2)
$$

$$
\times R'_{21}(0) \left[g^3 f_{\mathrm{tree}}(k_1, k_2) + g^5 f_{\mathrm{loop}}(k_1, k_2) \right]
$$

$$
+\text{permutations}. \tag{3.2}
$$

The functions $f_{\mathrm{tree}}(k_1, k_2)$ and $f_{\mathrm{loop}}(k_1, k_2)$ arise from the tree graph and the loop graph, respectively and read

$$
f_{\mathrm{tree}}(k_1, k_2) = \frac{2\sqrt{2}}{3m^{3/2} k_1 k_2^3}
$$

$$
\times \left(8E^2 + 12|E|k_2 + 3k_2^2 - 8\sqrt{|E|}(|E| + k_2)^{3/2} \right),
$$

$$
f_{\mathrm{loop}}(k_1, k_2) = \frac{1}{2\pi m^2} \frac{1}{\left(\sqrt{|E|} + \sqrt{k_2 + |E|} \right)^3}. \tag{3.3}
$$

In the calculation of the width we use the gluon energies k_1 and k_2 as independent variables in the phase space integral, the region of integration is given by the triangle

$$
0 \leq k_1, k_2 \leq m - |E|/2 \approx m,
$$

$$
k_1 + k_2 \geq m - |E|/2 \approx m. \tag{3.4}
$$

As indicated, one can neglect the ultrasoft binding energy in the boundaries.

One next has to square the T-matrix in (3.2), perform the summation over gluon polarizations, and integrate over k_1 and k_2. Of course one can omit the permutations in one factor and, instead, multiply by a factor 6. The integration over k_1 gives a factor k_2 from the boundaries of phase space. Therefore only those terms are sensitive to the binding energy E where both factors become small for small k_2 (see the structure of (2.19)). For the tree term this is only the case in two permutations, namely the identity and the exchange $k_1 \leftrightarrow k_3$, in the remaining four one may drop E and perform both integrations analytically. In the two potentially "dangerous" terms, one has to keep the binding energy. The k_2-integration can be performed analytically (with the help of Mathematica). The remaining k_1-dependent function can then be split in two terms. The first term can be integrated analytically and contains the logarithmic dependence on E, while in the second term one can drop E and subsequently also perform the k_1-integration analytically. In the square of the non-abelian term, as well as in the mixed term, only the identity and the permutation $k_1 \leftrightarrow k_3$ contribute. One may put $E = 0$ there, no logarithmic singularity arises from these terms.

For positronium, where the non-abelian contribution is absent, the result for the decay width into three gammas finally becomes

$$
\Gamma(^1P_1^{+-} \to 3\gamma)
$$

$$
= \frac{8}{\pi} \frac{\alpha^3}{m^4} |R'_{21}(0)|^2 \left[\ln \frac{m}{|E|} - \left(\frac{25}{9} + \frac{\pi^2}{64} + 2\ln 2 \right) \right]
$$

$$= \frac{\alpha^8 m}{96\pi} \left[\ln \frac{16}{\alpha^2} - \left(\frac{25}{9} + \frac{\pi^2}{64} + 2\ln 2 \right) \right]. \qquad (3.5)$$

The width for the quarkonium decay into three gluons becomes (5/18 is the color factor)

$$\Gamma(^1P_1^{+-} \to 3 \text{ gluons}) = \frac{5}{18} \frac{8}{\pi} \frac{\alpha_{\mathrm{s}}^3}{m^4} |R'_{21}(0)|^2 \qquad (3.6)$$

$$\times \left[\ln \frac{m}{|E|} - \left(\frac{25}{9} + \frac{\pi^2}{64} + 2\ln 2 \right) + \frac{3\sqrt{3\pi}}{2\sqrt{2}} + \frac{3\pi}{8} \right].$$

Let us first discuss the positronium case (3.5). Comparing with the formula of Alekseev [5] which reads

$$\Gamma(^1P_1^{+-} \to 3\gamma)_{\text{Alekseev}} = \frac{\alpha^8 m}{96\pi} \ln \frac{32}{\alpha^2}, \qquad (3.7)$$

we see that the logaritmic term $\ln 16/\alpha^2$ coincides, but the constant differs. Instead of $\ln 2 = 0.693$ in Alekseev's formula we have $-\left(\frac{25}{9} + \frac{\pi^2}{64} + 2\ln 2 \right) = -4.318$. The reason for the discrepancy appears to be that the author of [5] dropped terms of order unity compared to the logarithm. Therefore his constant should be only considered as giving the correct order of magnitude. The precise value of the constant is, however, of importance even in QED where the logarithm dominates, as seen from a comparison of the numerical result. The lifetime according to our formula becomes $\tau = 0.00582\,\mathrm{s}$, which is a factor of 1.6 larger than the lifetime which is derived from (3.7).

There is also a connection to another interesting problem of QED. Recently Manohar and Ruiz-Femenía [6] reanalyzed the low energy photon spectrum of the three photon decay of orthopositronium in the framework of NRQCD. This work was motivated by the observation of Pestieau and Smith [7] that the spectrum, as derived by Ore and Powell [8] long ago, violates Low's theorem [9]. The authors found agreement with the Ore–Powell formula for the case that all photons are hard, but obtained deviations if one photon becomes soft. The low energy photon spectrum (the energies of the remaining two photons are integrated) does not behave like the Ore–Powell spectrum but is much softer.

In our approach it is easy to understand the origin of such a suppression. The k has been replaced by $k + |E| + \mathbf{p}^2/m$ in some of the denominators, i.e. for $k \to 0$ there is a suppression factor. The above replacement is, however, only relevant in the denominators which can become small, not in those which are always hard. In order to obtain the correct spectrum one must therefore redo the whole Ore–Powell calculation from the beginning. This will not be done here, but it would be interesting to check whether we reproduce the results of [6].

We next discuss our result (3.6) for the quarkonium decay into three gluons. As far as the logarithmic term is concerned it coincides with the formula given by Barbieri, Gatto, and Remiddi [1]. Beyond this we were able to calculate the associated constant term to be added. The three contributions arise from the square of the tree graph, the mixed term, and the square of the loop graph. Numerically

they are

$$-4.31828 + 3.25621 + 1.17810 = 0.116018.$$

This is a rather striking result. Although the individual constants are rather large, they almost cancel in the sum. To get a feeling for the importance of the constant term let us use $|E|/m = (4\alpha_{\mathrm{s}}/3)^2/16$ and choose $\alpha_{\mathrm{s}} = 0.2$ which gives $m/|E| = 225$ and $\ln(m/|E|) = 5.4161$. This shows that, without the consideration of the non-abelian contributions, the constant term would be quite important. Even worse, with the constant -4.318 from the tree graph alone the whole expression would only stay positive if $m/|E| > 75$, and the whole approach would make sense only for rather small binding energies.

Some remarks on the error are appropriate. It is a term of the order $\sqrt{|E|/m}$ in the brackets of (3.5) and (3.6). A consistent calculation of this correction would be extremely cumbersome, considering the many places where we have dropped the binding energy. It is, however, possible to calculate the correction coming from the last step, when performing the integration in the two permutations which lead to the logarithmic term. It results in the addition of a term $13.1\sqrt{|E|/m} \approx 0.87$ in the bracket of (3.6). The correction is thus not dramatic, in spite of the quare root behavior and the large factor in front.

One might be tempted to simplify the whole procedure by replacing the non-local $\mathbf{p}^2/m$-dependence in (2.18) and (2.19) by some average $\overline{\mathbf{p}^2/m}$. This would directly lead to a result involving $|R'(0)|^2$ for any wave function. In the non-abelian loop graph there is a dependence on the variable $(\mathbf{p} - \mathbf{q})^2$, and it is hard to find a reasonable approximation for this. Therefore we restrict the discussion to the abelian case. The result would be

$$\Gamma(^1P_1^{+-} \to 3\gamma) \qquad (3.8)$$

$$\approx \frac{\alpha^8 m}{96\pi} \left[\ln \frac{16}{\alpha^2} - \left(\frac{3}{4} + \frac{\pi^2}{64} + \ln(1 + \overline{\mathbf{p}^2/(m|E|)}) \right) \right].$$

One could next use the virial theorem for the Coulomb potential and put $\overline{\mathbf{p}^2/m} = |E|$. This would obviously lead to a constant different from that in (3.5). The width would be multiplied by a factor of 1.33 as compared to (3.5). Thus one obtains a bad approximation even in QED where the logarithmic term $\ln(1/\alpha^2)$ dominates. This result shows that the non-locality is essential. On the other hand it demonstrates the stability of the logarithmic term $\sim \ln(1/\alpha^2)$ against any reasonable approximation which respects the fact that $\mathbf{p}^2/m$ and E are of the same order.

4 Conclusions

To our knowledge there are no data on the decay of the singlet P positronium state. Unfortunately at present there is also no possibility to test our results for quarkonia experimentally. For the $c\bar{c}$ system there is only a doubtful candidate [10] at 3526 MeV for the $^1P_1^{+-}$-state h_c, with width $< 1.1\,\mathrm{MeV}$ and no annihilation decays observed.

For the $b\bar{b}$ system there is not even a candidate. This is unfortunate because these states would not only provide an excellent further test for the nature of the long range spin dependent potential, but also for our ideas about annihilation decays. Our prediction for the annihilation width is only a slight modification of that given in [1] and reads

$$\frac{\Gamma(^1P_1^{+-})}{\Gamma(^3P_0^{++})} = \frac{10}{27}\frac{\alpha_{\rm s}}{\pi}\left(\ln\frac{m}{|E|} + 0.116\right). \qquad (4.1)$$

Note, however, that the small number to be added to the logarithm originates in a highly non-trivial way. It is a sum of three terms as derived in the previous section: $0.116 = -4.318 + 3.256 + 1.178$. The tree graph alone would give a large negative constant, possibly overcompensate the positive logarithm, and cause the whole approach to collapse.

From a purely theoretical point of view the annihilation decays under consideration are highly interesting and provided some puzzles in the past. We have shown here that a four dimensional treatment of the decay immediately leads to a result which is free of infrared diverences and which, furthermore, allows a definite calculation for any given wave function. Notwithstanding the fact that NRQCD has certainly greatly improved our understanding of low energy QCD and has led to a systematic and effective way for calculations, we believe that for some problems, like the one discussed here, a four dimensional treatment is more transparent and effective.

We derived our result by using the $Q\bar{Q}$ wave function only. Nevertheless it is easy to make contact with the three dimensional approach. The contributions from the poles in the T-matrix can, alternatively, be interpreted as contributions from $Q\bar{Q}g$ components (with a soft gluon) in the wave function. The advantage in our approach is that we obtain this contribution immediately in an explicit form. Obviously these two ways of viewing the situation would also apply to decays into an arbitrary number of gluons.

The decay of the spin triplet state $^3P_1^{++}$ into a real gluon plus a virtual gluon which further decays into a light quark–antiquark pair shows the same infrared problems as the decay discussed here. Phenomenologically it is more interesting, because the charmonium state $\chi_{c1}(1P)$ is experimentally well established and theoretical results can be compared with the data. This will be undertaken in a forthcoming work.

References

1. R. Barbieri, R. Gatto, E. Remiddi, Phys. Lett. 61 B, 465 (1976), Phys. Rev. D **51**, 1125 (1995), Erratum **55**, 5853 (1997)
2. G.T. Bodwin, E. Braaten, G.P. Lepage, Phys. Rev. D **46**, R1914 (1992); G.T. Bodwin, E. Braaten, T.C. Yuan, G.P. Lepage, Phys. Rev. D **46**, R3703 (1992); G.T. Bodwin, E. Braaten, G.P. Lepage, Phys. Rev. D **51**, 1125 (1995); Phys. Rev. D **55**, 5853(E) (1997)
3. W.E. Caswell, G.P. Lepage, Phys. Lett. B **167**, 437 (1986)
4. D. Gromes, Eur. Phys. J. C **30**, 233 (2003)
5. A.I. Alekseev, Soviet. Phys. JETP **9**, 1312 (1959); (J. Expt. Theoret. Phys. (U.S.S.R.) **36**, 1839 (1959)); see also the review of M.A. Stroscio, Phys. Rep. **22**, 215 (1975)
6. A.V. Manohar, P. Ruiz-Femenía, hep-ph/0311002
7. J. Pestieau, C. Smith, Phys. Lett. B **524**, 395 (2002)
8. A. Ore, J.L. Powell, Phys. Rev. **75**, 1696 (1949)
9. F.E. Low, Phys. Rev. **110**, 974 (1958)
10. K. Hagiwara et al. (Particle Data Group), Phys. Rev. D **66**, 010001 (2002) and 2003 partial update for edition 2004 (http://pdg.lbl.gov)
11. R. Barbieri, R. Gatto, R. Kögerler, Phys. Lett. B **60**, 183 (1976)